

REFINEMENT OF SOME GENERALIZED POLYNOMIAL INEQUALITIES

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Abstract: Let $p(z)$ be a polynomial of degree n , with $p(z) = \sum_{\nu=0}^n a_{\nu} z^{\nu}$. In this paper, firstly, we prove a result for bounds in two different radii for the maximum modulus of lacunary type of polynomials having all zeros outside a disk of given radius. Next, we establish a result concerning the maximum modulus of polar derivative of polynomial having all zeros inside a disk out of which some fixed number of zeros are centred at origin. In particular case this result reduces into ordinary derivative. Our results not only provide refinements of earlier proved results but also open new avenues for other results in the same field of research.

Keywords and Phrases: Polynomials; Derivatives; Polar Derivative; Inequalities; Maximum Modulus; Zeros.

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1. Introduction and Statement of Results

If $p(z)$ be a polynomial of degree n and $p'(z)$ its derivative. Then according to a well known result due to S. Bernstein [3], we have

$$\max_{|z|=1} |p'(z)| \leq n \max_{|z|=1} |p(z)|. \quad (1.1)$$

The result is best possible and equality holds for $p(z) = \alpha z^n$, where $|\alpha| = 1$.

If we restrict to the class of polynomials having no zero in $|z| < 1$, then the bound in inequality (1.1) can be improved. In this direction, P. Erdős conjectured and later Lax [9] verified that if $p(z) \neq 0$ in $|z| < 1$, then

$$\max_{|z|=1} |p'(z)| \leq \frac{n}{2} \max_{|z|=1} |p(z)|. \quad (1.2)$$

On the other hand if a polynomial $p(z)$ of degree n has all its zeros in $|z| \leq 1$. Then it was shown by Turan [14], that

$$\max_{|z|=1} |p'(z)| \geq \frac{n}{2} \max_{|z|=1} |p(z)|. \quad (1.3)$$

Thus in (1.2) as well as in (1.3) equality holds for those polynomials of degree n which have all their zeros on $|z| = 1$. By simple deduction from the maximum modulus principle (for instance see [10] or [12]) for a polynomial $p(z)$ of degree n and for every $R \geq 1$, we have

$$\max_{|z|=R} |p(z)| \leq R^n \max_{|z|=1} |p(z)|. \quad (1.4)$$

The result is best possible and extremal polynomial is $p(z) = \lambda z^n$, $\lambda (\neq 0)$ is a complex number. For the class of polynomials having no zeros in $|z| < 1$, the inequality (1.4) is sharpened by Ankeny and Rivlin [1] as

$$\max_{|z|=R \geq 1} |p(z)| \leq \frac{R^n + 1}{2} \max_{|z|=1} |p(z)|. \quad (1.5)$$

Corresponding inequalities under the same conditions as in inequalities (1.4) and (1.5) with $r \leq 1$ were proved by Zerrantonello and Varga [15] and Rivlin [16] respectively.

For a polynomial $p(z)$ of degree n , $D_\alpha p(z)$, the polar derivative of $p(z)$ with respect to the point α is defined as

$$D_\alpha p(z) = np(z) + (\alpha - z)p'(z).$$

It is easy to see that $D_\alpha p(z)$ is a polynomial of degree at most $n - 1$ and that $D_\alpha p(z)$ generalizes the ordinary derivative in the sense that

$$\lim_{\alpha \rightarrow \infty} \left[\frac{D_\alpha p(z)}{\alpha} \right] = p'(z).$$

For the polynomial $p(z)$ of degree n having all its zeros in $|z| < k, k > 0$ and for $r \leq k \leq R$, Jain [7] proved

$$\frac{M(p, r)}{r^n + kr^{n-1}} \geq \frac{M(p, R)}{R^n + kR^{n-1}}, \quad (1.6)$$

where $M(p, R) = \max_{|z|=R} |p(z)|$. Here and hereafter we shall also mean $M(p, R) = \max_{|z|=R} |p(z)|$. Inequality (1.6) is generalized by Keshtwal [8] for lacunary type of polynomials, as following.

Theorem 1.1. [Keshtwal [8]] *If $p(z) = a_0 + \sum_{j=\mu}^n a_j z^j$ is a polynomial of degree n , having all its zeros in $|z| > k, k > 1$, then for $r \leq k \leq R$*

$$\begin{aligned} & \frac{M(p, R)}{R^{n-1}(R^{\mu+1} + k^{\mu+1})n|a_0| + \mu|a_\mu|R^n k^{\mu+1}(R^{\mu-1} + k^{\mu-1})} \\ & \leq \frac{M(p, r)}{r^{n-1}(r^{\mu+1} + k^{\mu+1})n|a_0| + \mu|a_\mu|r^n k^{\mu+1}(r^{\mu-1} + k^{\mu-1})}. \end{aligned} \quad (1.7)$$

The following result concerning polar derivative of polynomial is due to Govil and Kumar [5].

Theorem 1.2. [Govil and Kumar [5]] *If $p(z) = z^s(a_0 + a_1 z^1 + a_2 z^2 + \dots + a_{n-s} z^{n-s}), 0 \leq s \leq n$ is polynomial of degree n having all its zeros in $|z| \leq k, k \geq 1$, then for any complex number α with $|\alpha| \geq k$*

$$\max_{|z|=1} |D_\alpha p(z)| \geq (|\alpha| - k) \left(\frac{n+s}{1+k^n} + \frac{|a_{n-s}|k^{n-s} - |a_0|}{(1+k^n)(|a_{n-s}|k^{n-s} + |a_0|)} \right) \max_{|z|=1} |p(z)|. \quad (1.8)$$

In this paper, firstly we are able to improve upon the Theorem 1.1 by proving the following.

Theorem 1.3. *If $p(z) = a_0 + \sum_{j=\mu}^n a_j z^j$ is a polynomial of degree n , having all its*

zeros in $|z| > k, k > 1$, then for $r \leq k \leq R$

$$\frac{M(p, R)}{f(R, k)} \leq \frac{M(p, r)}{f(r, k)} - \frac{(R^n - r^n)(n|a_0|k^{\mu+1} + \mu|a_\mu|r k^{2\mu})}{k^n f(R, k)(r^{\mu+1} + k^{\mu+1})n|a_0| + \mu|a_\mu|(r^\mu k^{\mu+1} + r k^{2\mu})} m(p, k), \quad (1.9)$$

where $m = m(p, k) = \min_{|z|=k} |p(z)|$ and $f(R, k) = R^{n-1}(R^{\mu+1} + k^{\mu+1})n|a_0| + \mu|a_\mu|R^n k^{\mu+1}(R^{\mu-1} + k^{\mu-1})$ etc. Here and hereafter we shall also mean $m = m(p, k) = \min_{|z|=k} |p(z)|$.

Next, we improve upon the Theorem 1.2 as following.

Theorem 1.4. If $p(z) = z^s(a_0 + a_1 z^1 + a_2 z^2 + \dots + a_{n-s} z^{n-s}), 0 \leq s \leq n$ is polynomial of degree n having all its zeros in $|z| \leq k, k \geq 1$, then for any complex number α with $|\alpha| \geq k$

$$\max_{|z|=1} |D_\alpha p(z)| \geq \frac{(|\alpha| - k)}{1 + k^n} \left((n + s) + \frac{|a_{n-s}|k^{n-s} - |a_0|}{|a_{n-s}|k^{n-s} + |a_0|} \right) \left\{ \max_{|z|=1} |p(z)| + \frac{k^{n-1}}{2k^n} \min_{|z|=1} |p(z)| \right\}. \quad (1.10)$$

If we divide both sides of inequality (1.10) by $|\alpha|$ and then let $|\alpha| \rightarrow \infty$, then we obtain the following corollary which is of independent interest too.

Corollary 1.1. If $p(z) = z^s(a_0 + a_1 z^1 + a_2 z^2 + \dots + a_{n-s} z^{n-s}), 0 \leq s \leq n$ is polynomial of degree n having all its zeros in $|z| \leq k, k \geq 1$, then

$$\max_{|z|=1} |p'(z)| \geq \frac{1}{1 + k^n} \left((n + s) + \frac{|a_{n-s}|k^{n-s} - |a_0|}{|a_{n-s}|k^{n-s} + |a_0|} \right) \left\{ \max_{|z|=1} |p(z)| + \frac{k^{n-1}}{2k^n} \min_{|z|=1} |p(z)| \right\}. \quad (1.12)$$

Corollary 1.1 is an improvement of a result due to Govil and Kumar ([5], Corollary 1.2).

2. Lemmas

To prove the main results, we need the following lemmas.

Lemma 2.1. [Govil, Rahman and Schmeisser [6]] If $p(z) = \sum_{j=0}^n a_j z^j$ is a polynomial of degree n having no zeros in $|z| \leq k, k \geq 1$, then

$$\max_{|z|=1} |p'(z)| \leq n \frac{n|a_0| + k^2|a_1|}{(1 + k^2)n|a_0| + 2k^2|a_1|} \max_{|z|=1} |p(z)|. \quad (2.1)$$

The result is best possible with extremal polynomial $p(z) = (z + k)^n$.

Qazi [11] generalized the Lemma 2.1 and proved the following.

Lemma 2.2. [Qazi [11]] If $p(z) = a_0 + \sum_{j=\mu}^n a_j z^j$, $1 \leq \mu \leq n$ is a polynomial of degree n not vanishing in $|z| \leq k$, $k \geq 1$, then

$$\max_{|z|=1} |p'(z)| \leq n \frac{1 + \frac{\mu}{n} \left| \frac{a_\mu}{a_0} \right|}{1 + k^{\mu+1} + \frac{\mu}{n} \left| \frac{a_\mu}{a_0} \right| (k^{\mu+1} + k^{2\mu})} \max_{|z|=1} |p(z)|. \quad (2.2)$$

Lemma 2.3. If $p(z) = a_0 + \sum_{j=\mu}^n a_j z^j$, $1 \leq \mu \leq n$ is a polynomial of degree n not vanishing in $|z| \leq k$, $k \geq 1$, then

$$\begin{aligned} \max_{|z|=1} |p'(z)| &\leq n \frac{n|a_0| + \mu|a_\mu|k^{\mu+1}}{(1 + k^{\mu+1})n|a_0| + \mu|a_\mu|(k^{\mu+1} + k^{2\mu})} \|p\| \\ &\quad - \frac{n}{k^n} \left[1 - \frac{n|a_0| + \mu|a_\mu|k^{n+1}}{(1 + k^{\mu+1})n|a_0| + \mu|a_\mu|(k^{\mu+1} + k^{2\mu})} \right] m, \end{aligned} \quad (2.3)$$

where $\|p\| = \max_{|z|=1} |p(z)|$ and $m = m(p, k) = \min_{|z|=k} |p(z)|$.

Proof. Since $p(z)$ does not vanish in $|z| \leq k$, $k \geq 1$ and $|p(z)| \geq m = \min_{|z|=k} |p(z)|$, therefore by Rouché's theorem, the polynomial $F(z) = p(z) + \lambda \frac{z^n}{k^n} m$, $|\lambda| < 1$, also does not vanish in $|z| \leq k$, $k \geq 1$. On applying inequality (2.2) of Lemma 2.2 to the polynomial $F(z) = p(z) + \lambda \frac{z^n}{k^n} m$, we have

$$\max_{|z|=1} |F'(z)| \leq n \frac{1 + \frac{\mu}{n} \left| \frac{a_\mu}{a_0} \right|}{1 + k^{\mu+1} + \frac{\mu}{n} \left| \frac{a_\mu}{a_0} \right| (k^{\mu+1} + k^{2\mu})} \max_{|z|=1} |F(z)|$$

or

$$\begin{aligned} \max_{|z|=1} |p'(z) + \lambda n \frac{z^{n-1}}{k^n} m| &\leq n \frac{1 + \frac{\mu}{n} \left| \frac{a_\mu}{a_0} \right|}{1 + k^{\mu+1} + \frac{\mu}{n} \left| \frac{a_\mu}{a_0} \right| (k^{\mu+1} + k^{2\mu})} \max_{|z|=1} |p(z) + m \lambda \frac{z^n}{k^n}| \\ &\leq n \frac{1 + \frac{\mu}{n} \left| \frac{a_\mu}{a_0} \right|}{1 + k^{\mu+1} + \frac{\mu}{n} \left| \frac{a_\mu}{a_0} \right| (k^{\mu+1} + k^{2\mu})} \left\{ \max_{|z|=1} |p(z)| + \frac{m|\lambda|}{k^n} \right\}. \end{aligned} \quad (2.4)$$

Now, since λ is on our choice, choosing argument of λ in such a way that L.H.S. of inequality (2.4) becomes

$$\max_{|z|=1} |p'(z) + \lambda n \frac{z^{n-1}}{k^n} m| = \max_{|z|=1} |p'(z)| + \frac{|\lambda|mn}{k^n}. \quad (2.5)$$

Inequality (2.4) in association with (2.5) gives

$$\max_{|z|=1} |p'(z)| + \frac{|\lambda|mn}{k^n} \leq n \frac{1 + \frac{\mu}{n} \left| \frac{a_\mu}{a_0} \right|}{1 + k^{\mu+1} + \frac{\mu}{n} \left| \frac{a_\mu}{a_0} \right| (k^{\mu+1} + k^{2\mu})} \left\{ \max_{|z|=1} |p(z)| + \frac{m|\lambda|}{k^n} \right\}. \quad (2.6)$$

Inequality (2.6) on simplification gives

$$\begin{aligned} \max_{|z|=1} |p'(z)| &\leq n \frac{n|a_0| + \mu|a_\mu|k^{n+1}}{(1 + k^{\mu+1})n|a_0| + \mu|a_\mu|(k^{\mu+1} + k^{2\mu})} \max_{|z|=1} |p(z)| \\ &\quad - \frac{n|\lambda|}{k^n} \left[1 - \frac{n|a_0| + \mu|a_\mu|k^{n+1}}{(1 + k^{\mu+1})n|a_0| + \mu|a_\mu|(k^{\mu+1} + k^{2\mu})} \right] m. \end{aligned} \quad (2.7)$$

Finally, letting $|\lambda| \rightarrow 1$ in (2.7), we get the desired inequality (2.3) and thus the proof of the Lemma 2.3 is completed.

Lemma 2.4. [Aziz [2]] *If $p(z)$ is a polynomial of degree n which has all its zeros in the disk $|z| \leq k, k \geq 1$, then*

$$\max_{|z|=k} |p(z)| \geq \frac{2k^n}{1 + k^n} \max_{|z|=1} |p(z)|. \quad (2.8)$$

This result also independently proved by Govil and Kumar [3]. Further, Lemma 2.4 was improved by Dewan et.al.([4], Inequality (20)) as

Lemma 2.5. *If $p(z)$ is a polynomial of degree n which has all its zeros in the disk $|z| \leq k, k \geq 1$, then*

$$\max_{|z|=k} |p(z)| \geq \frac{2k^n}{1 + k^n} \max_{|z|=1} |p(z)| + \frac{k^n - 1}{k^n + 1} \min_{|z|=k} |p(z)|. \quad (2.9)$$

3. Proof of the Main Theorems

Proof of Theorem 1.3. Let $0 < r \leq k$. Since $p(z)$ has all its zeros in $|z| \geq k, k \geq 1$, therefore the polynomial $T(z) = p(rz)$ has all its zeros in $|z| \geq \frac{k}{r}, \frac{k}{r} \geq 1$. Applying Lemma 2.3 to the polynomial $T(z)$, we get

$$\begin{aligned} \max_{|z|=1} |T'(z)| &\leq n \frac{n|a_0| + \mu|a_\mu| \left(\frac{k}{r}\right)^{\mu+1}}{(1 + \left(\frac{k}{r}\right)^{\mu+1})n|a_0| + \mu|a_\mu| \left\{ \left(\frac{k}{r}\right)^{\mu+1} + \left(\frac{k}{r}\right)^{2\mu} \right\}} \max_{|z|=1} |T(z)| \\ &\quad - \frac{n}{\left(\frac{k}{r}\right)^n} \left[1 - \frac{n|a_0| + \mu|a_\mu| \left(\frac{k}{r}\right)^{n+1}}{(1 + \left(\frac{k}{r}\right)^{\mu+1})n|a_0| + \mu|a_\mu| \left\{ \left(\frac{k}{r}\right)^{\mu+1} + \left(\frac{k}{r}\right)^{2\mu} \right\}} \right] \min_{|z|=\frac{k}{r}} |T(z)| \end{aligned}$$

which implies

$$\max_{|z|=1} r|p'(rz)| \leq nA \max_{|z|=1} |p(rz)| - \frac{nr^n}{k^n} (1-A) \min_{|z|=\frac{k}{r}} |p(rz)|$$

or

$$\max_{|z|=r} |p'(z)| \leq \frac{n}{r} A \max_{|z|=r} |p(z)| - \frac{nr^{n-1}}{k^n} (1-A) \min_{|z|=k} |p(z)|, \quad (3.1)$$

where

$$A = \frac{n|a_0|r^{\mu+1} + \mu|a_\mu|k^{\mu+1}r^\mu}{(r^{\mu+1} + k^{\mu+1})n|a_0| + \mu|a_\mu|(k^{\mu+1}r^\mu + rk^{2\mu})}. \quad (3.2)$$

As $p'(z)$ is a polynomial of degree at most $n-1$, then by maximum modulus principle ([10], p. 158, problem III 269), we have

$$\frac{M(p', t)}{t^{n-1}} \leq \frac{M(p', r)}{r^{n-1}}, \quad \text{for } t \geq r. \quad (3.3)$$

Combining inequalities (3.1) and (3.3), we have

$$M(p', t) \leq \frac{t^{n-1}}{r^{n-1}} \left(\frac{n}{r} A \max_{|z|=r} |p(z)| - \frac{nr^{n-1}}{k^n} (1-A) \min_{|z|=k} |p(z)| \right).$$

Now, we have for $0 \leq \theta < 2\pi$

$$\begin{aligned} |p(Re^{i\theta}) - p(re^{i\theta})| &\leq \int_r^R |p'(te^{i\theta})| dt \\ &\leq \int_r^R \frac{nt^{n-1}}{r^{n-1}} \left(\frac{A}{r} \max_{|z|=r} |p(z)| - \frac{r^{n-1}}{k^n} (1-A) \min_{|z|=k} |p(z)| \right) dt \\ &= A \frac{(R^n - r^n)}{r^n} M(p, r) - (1-A) \frac{(R^n - r^n)}{k^n} m(p, k). \end{aligned} \quad (3.4)$$

From inequality (3.4) it implies that

$$|p(Re^{i\theta})| \leq \left[1 + \frac{(R^n - r^n)}{r^n} A \right] M(p, r) - \frac{(R^n - r^n)}{k^n} (1-A) m(p, k). \quad (3.5)$$

Substituting expression for A from (3.2) in (3.5) and after simplification together with using the fact that $R \geq r$, (3.5) reduces to

$$M(p, r) \leq \left[\frac{R^{n-1}(R^{\mu+1} + k^{\mu+1})n|a_0| + \mu|a_\mu|R^n k^{\mu+1}(R^{\mu-1} + k^{\mu-1})}{r^{n-1}(r^{\mu+1} + k^{\mu+1})n|a_0| + \mu|a_\mu|r^n k^{\mu+1}(r^{\mu-1} + k^{\mu-1})} \right] M(p, r) \\ - \frac{(R^n - r^n)}{k^n} \left(1 - \frac{n|a_0|r^{\mu+1} + \mu|a_\mu|k^{\mu+1}r^\mu}{(r^{\mu+1} + k^{\mu+1})n|a_0| + \mu|a_\mu|(k^{\mu+1}r^\mu + rk^{2\mu})} \right) m(p, k).$$

This is ultimately equivalent to inequality (1.9) and thus proof of the Theorem 1.3 is completed.

Proof of Theorem 1.4. The proof of the Theorem 1.4 runs parallel to the proof given by Govil and Kumar (see [5] proof of Theorem 1.1). Here we apply the improved Lemma 2.5 instead of Lemma 2.4. So here we skip the details.

4. Conclusion

In this paper, we are able to get inequalities that provide better bounds for maximum modulus of polynomial functions in different radii and also for the maximum modulus of polar and ordinary derivatives of the polynomials. The limitation of our results in this paper is that these do not give bound for the values of $k \in (0, 1)$. The scope of the bounds for $k \in (0, 1)$ is still open and recommended for further research.

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